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Second Note on Weierstrass' Theory of Elliptic Functions.

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Origin of the Periodic Functions.

Instead of beginning his investigations by any reference to the elliptic integrals as did Legendre, Abel and Jacobi, Weierstrass seeks the origin of the periodic functions in a fundamental function-theoretical problem, namely: a series of argument values following a given analytical law inserted successively in a single valued analytic function of one variable gives rise to a series of corresponding functional values connected by a law which depends upon the character of the function when that is supposed given, but when the law connecting the argument values and a second law connecting the corresponding functional values are given, then the general character of the function is thereby determined, and, retaining either of the given laws, any class of functions whatever can be defined by a suitable choice of the other law. We may choose, for example, to let the successive argument-values form an arithmetic series, so that, u_1, u_2, u_3 being any three consecutive values, the analytical law connecting them is represented by $2u_2 = u_1 + u_3$. If now the corresponding functional values form a geometric series, we arrive easily at the exponential function as the solution, and if the two laws be exchanged we shall have defined, as might have been expected, the converse function or

$$\int_1 \frac{dx}{x}.$$

Retaining for the series of argument-values the above law $2u_2 = u_1 + u_3$, I shall show that, if the functional values are connected by a law expressible as a rational integral function, we shall have defined the single valued doubly-periodic functions. In other words Weierstrass starts from the addition-theorem.

Let $\phi(u)$ be a single valued analytic function possessing the last named property, it must, therefore, be developable in a series arranged according to the positive and ascending powers of $u - a$, where a is not a singular point for ϕu. I shall call this a "power series of $(u - a)$," namely,

$$\phi u = A_0 + A_1 (u - a) + A_2 (u - a)^2 + \ldots$$

Substitute for u the values successively, u_1, u_2, u_3, where $2u_2 = u_1 + u_3$,

$$\phi (u_1) - A_0 = A_1 (u_1 - a) + A_2 (u_1 - a)^2 + \ldots$$
$$\phi (u_2) - A_0 = A_1 (u_2 - a) + A_2 (u_2 - a)^2 + \ldots$$
$$\phi (u_3) - A_0 = A_1 (u_3 - a) + A_2 (u_3 - a)^2 + \ldots$$

We can choose a so that $A_1 \gtrless 0$, and the series will converge and accurately represent the value of the function if only mod. $(u_1 - a)$ and mod. $(u_3 - a)$ be taken sufficiently small, since mod. $u_2 - a$ will then of necessity be within the circle of convergence. By inverting the series we can therefore develop $(u - a)$ in a power series of $\phi u - A_0$, which series will converge so long as mod. $(\phi u - A_0)$ be kept within corresponding limits. We are justified therefore in writing

$$u_1 - a = \frac{1}{A_1} (\phi u_1 - A_0) + \ldots$$
$$u_2 - a = \frac{1}{A_1} (\phi u_2 - A_0) + \ldots$$
$$u_3 - a = \frac{1}{A_1} (\phi u_3 - A_0) + \ldots$$

Since now $u_2 = \dfrac{u_1 + u_3}{2}$ we have

$$u_2 - a = \frac{1}{A_1} (\phi u_2 - A_0) + \ldots = \frac{1}{A_1} \left(\frac{\varphi u_1 + \varphi u_3}{2} - A_0 \right) + \ldots$$

which establishes a relation between ϕu_1, ϕu_2, ϕu_3,

$$\text{or } F(\phi u_1,\ \phi u_2,\ \phi u_3) = 0.$$

I shall examine that class of functions where F represents a rational integral function. This addition equation can be shown in another form. So long as $u'_1 + u'_3 = u_1 + u_3$, then $u_2 = u'_2$ and the equation

$$F(\phi u'_1,\ \phi u_2,\ \phi u'_3) = 0$$

will also hold. Eliminating ϕu_2 we obtain

$$G(\phi u_1,\ \phi u_3,\ \phi u'_1,\ \phi u'_3) = 0.$$

Choosing u_1 and u_3 so that mod. $(u_1 + u_3)$ is within the limit of convergence, we can make $u'_3 = 0$, whence $u'_1 = u_1 + u_3$ and the last equation becomes

$$G(\phi u_1,\ \phi u_3,\ \phi (u_1 + u_3)) = 0.$$

By differentiating first with respect to ϕu_1 and then with respect to ϕu_3 we arrive at another form still of the addition equation. Writing for convenience instead of ϕu_1, ϕu_3, $\phi (u_1 + u_3)$; x, y, z, respectively, and instead of u_3, v, we have

$$G(x,\ y,\ z) = 0,$$

$$\frac{\partial G}{\partial x}\,\frac{\partial x}{du}+\frac{\partial G}{\partial z}\,\frac{dz}{du}=0,$$

$$\frac{\partial G}{\partial y}\,\frac{dy}{dv}+\frac{\partial G}{\partial z}\,\frac{dz}{dv}=0,$$

whence

$$\frac{\partial G}{\partial x}\,\frac{dx}{du}-\frac{\partial G}{\partial y}\,\frac{dy}{dv}=0;$$

which last equation combined with the first gives

$$G_1\left(x,\ y,\ \frac{dx}{du},\ \frac{dy}{dv}\right)=0,$$

which does not contain z. If now we assign to $v=u_3$ a constant value, the last equation yields an equation between x and $\frac{dx}{du}$, for if G_1 should vanish identically, that is if every coefficient of $x^\lambda\left(\frac{dx}{du}\right)^\mu$ should vanish, that would give a relation between y and $\frac{dy}{dv}$ which amounts to the same thing. The two equations

$$G(x,\ y,\ z)=0,$$

$$\frac{\partial G}{\partial x}\,\frac{dx}{du}-\frac{\partial G}{\partial y}\,\frac{dy}{dv}=0$$

must have at least one common root, therefore at least one common factor which is a linear function of z and a rational integral function of the coefficients of z in the two equations; therefore $z=R\left(x,\ y,\ \frac{dx}{du},\ \frac{dy}{dv}\right)$, that is to say, $\phi(u+v)=R(\phi u,\ \phi v,\ \phi' u,\ \phi' v)$, where R is a rational function.

If $\phi(u+v)$ be developable in a power series which converges for mod. $u<\frac{r}{2}$ and mod. $v<\frac{r}{2}$, then if we write $u=v$, the function

$$\phi(2u)=R(\phi u,\ \phi' u)$$

being a rational function is representable as the quotient of two integral functions, and

$$\phi u=R\left(\phi\frac{u}{2},\ \phi'\frac{u}{2}\right)=\frac{f_0\left(\phi\frac{u}{2},\ \phi'\frac{u}{2}\right)}{g_0\left(\phi\frac{u}{2},\ \phi'\frac{u}{v}\right)}$$

or

$$\phi u=\frac{f_1 u}{g_1 u},$$

where f_1 and g_1 converge for mod. $u<2r$. In like manner

$$\phi\frac{u}{2}=\frac{f_1\frac{u}{2}}{g_1\frac{u}{2}}=\frac{f_2 u}{g_2 u},$$

where f_2 and g_2 converge for mod. $u < 4r$, and proceeding in this manner, the function ϕu can be represented as the quotient of two functions fu and gu with infinitely large circles of convergence. But since the quotient coincides with ϕu for all values of u such that mod. $u < r$, therefore it is the analytical continuation of ϕu, and represents the same function. Since ϕu is represented as the quotient of two power series which converge for all finite values of the argument, therefore ϕu has no essential singularities unless it be at infinity. These two functions, fu, gu, are either rational or transcendental integral functions. Whether they are rational or transcendental depends upon the number and kind of singular points which the function ϕu possesses. The case where both fu and gu are rational does not interest us. When fu is transcendental and gu rational then ϕu has one essential singularity at infinity and a finite number of non-essential singularities or poles. When, finally, both fu and gu are transcendental, the number of polar singularities is infinite, although there is still only a finite number of them in any finite region. When one or both of the functions fu, gu, are transcendental, then ϕu is periodic, for, being transcendental, the equation

$$\phi v - b = 0$$

is satisfied for as many different values of v as one pleases, for example, for $n+1$ values, so that $$\phi v_1 = \phi v_2 = \ldots \phi v_{n+1} = b.$$

Let the equation $$g(\phi(u+v),\ \phi u,\ b) = 0$$

be of the n^{th} degree in $\phi(u+v)$, it can then be satisfied for only n different values of $\phi(u+v)$, whereby the only condition is mod. $u < r$. But it is satisfied for $n+1$ different values of v; therefore for at least two values of v we must have

$$\phi(u+v_\lambda) = \phi(u+v_\mu),$$

or, writing

$$v_\lambda - v_\mu = p,$$

$$\phi(u+p) = \phi(u),$$

and p is a period of the function. Any multiple of p is also a period. The smallest period then gives rise by its successive multiples to all the others of the class. If there is another class of periods, then they must be founded on a unit which is additively independent of the unit on which the first class is founded. But there are only two such units or rather, perhaps, unit-pairs. An analytic function of one variable cannot therefore have more than two classes of periods, because only two unit-pairs are possible so long as the fundamental laws of association and commutation hold. The Weierstrassian proof of this, which I venture to reproduce here, is as follows:

Let us suppose $e_1, e_2 \dots e_n$ to be n unit-pairs which are additively independent. We must also assume that the products of the units themselves are expressible as linear functions of the units. In order then that it may always be possible to carry out the operation of division, except when the divisor is zero, we should always be able to determine the coefficients of y in $xy = z$, when x and z are known and finite, and when

$$x = x_1 e_1 + x_2 e_2 + \dots x_n e_n$$
$$y = y_1 e_1 + y_2 e_2 + \dots y_n e_n$$
$$z = z_1 e_1 + z_2 e_2 + \dots z_n e_n$$
$$e_\lambda e_\mu = \alpha'_{\lambda\mu} e_1 + \alpha''_{\lambda\mu} e_2 + \dots \alpha^{(n)}_{\lambda\mu} e_n,$$

where $x_1, x_2 \dots x_n, y_1, y_2 \dots y_n, z_1, z_2 \dots z_n, \alpha'_{\lambda\mu}, \alpha''_{\lambda\mu} \dots \alpha^{(n)}_{\lambda\mu}$ are positive or negative real numbers. Owing to the independence of the unit-pairs as regards the notion of greater and less, we shall have, to determine the coefficients $y_1, y_2 \dots y_n$ of y, the n equations

$$(\alpha'_{11} x_1 + \alpha'_{21} x_2 + \dots \alpha'_{n1} x_n) y_1 + \dots + (\alpha'_{1n} x_1 + \alpha'_{2n} x_2 + \dots + \alpha'_{nn} x_n) y_n = z_1$$
$$(\alpha''_{11} x_1 + \alpha''_{21} x_2 + \dots \alpha''_{n1} x_n) y_1 + \dots + (\alpha''_{1n} x_1 + \alpha''_{2n} x_2 + \dots + \alpha''_{nn} x_n) y_n = z_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(\alpha^{(n)}_{11} x_1 + \alpha^{(n)}_{21} x_2 + \dots \alpha^{(n)}_{n1} x_n) y_1 + \dots + (\alpha^{(n)}_{1n} x_1 + \alpha^{(n)}_{2n} x_2 + \dots + \alpha^{(n)}_{nn} x_n) y_n = z_n$$

It will now be found impossible to transform the determinant so that it will not vanish for any system of values of $x_1, x_2 \dots x_n$ except the one

$$x_1 = x_2 = \dots = x_n = 0$$

if the number of unit-pairs is greater than two. If however $n = 2$ the determinant $\quad (\alpha'_{11} x_1 + \alpha'_{12} x_2)(\alpha''_{12} x_1 + \alpha''_{22} x_2) - (\alpha''_{11} x_1 + \alpha''_{12} x_2)(\alpha'_{12} x_1 + \alpha'_{22} x_2)$
must be and is expressible in the form

$$A x_1^2 + B x_2^2,$$

where A and B are positive. This determinant evidently does not vanish except for the single system of values $\quad x_1 = x_2 = 0$.
Except therefore when the divisor is zero, the operation of division can always be carried out with finite numbers based on one or on two, but not more than two pairs of units, retaining the associative and commutative laws of arithmetic. Analytic functions of one variable therefore, which possess an addition equation and have an infinite number of polar singularities can be displayed as quotients of two power-series which converge for all finite arguments, and are periodic, with one and not more than two fundamental pairs of periods.

Fixing the attention upon the circumstance that the function will possess an infinite number of polar singularities, or what amounts to the same thing, an infinite number of zero points, I shall proceed to determine a function by means of its zero points and by the fact of its convergence in general for all finite argument values.

ORIGIN OF THE SIGMA-FUNCTION.

If we seek a function gu such that

(1) $$g'_u = F\sum_\nu c_\nu u^\nu,$$

(2) $$g(0) = c,$$

the conditions can always be satisfied when the power-series $\sum_\nu c_\nu u^\nu$ converges, for then the series $\sum_\nu \frac{c_\nu}{\nu+1} u^{\nu+1}$ converges also, and both sides of (1) have the same radius of convergence. The function is easily seen to be $ce^{g(u)}$, where $g(0) = 0$, and gu means an entire function. But the following conditions will now be imposed in addition to the above. The function (3) shall vanish for $u = a_\nu$, $\nu = 0 \pm 1 \ldots \pm \infty$, with the condition lim. $a_\nu = \infty$, $(\nu = \infty)$, that is, the number of a's is infinite, but there shall be only a finite number of them in a finite region.

(4) $$G(b_\nu) \gtrless 0, \quad b_\nu \gtrless a_\nu,$$

(5) the function shall be developable in a power-series which converges for all finite values of the argument.

The last condition determines

$$Gu = C_\lambda (u-a)^\lambda + C_{\lambda+1}(u-a)^{\lambda+1} + \ldots$$

whence

$$G'u = \lambda C_\lambda (u-a)^{\lambda-1} + (\lambda+1)\, C_{\lambda+1}(u-a)^{\lambda-1} + \ldots$$

and

$$\frac{G'}{G} = \frac{\lambda + \frac{\lambda+1}{\lambda} C_\lambda (u-a) + \ldots}{(u-a)\left(1 + \frac{C_{\lambda+1}}{C_\lambda}(u-a) + \ldots\right)}$$

$$= \frac{\lambda}{u-a} + \Sigma(u-a)$$

where $\Sigma(u-a)$ means a power-series of $(u-a)$. The first logarithmic derivative of the function will therefore be essentially of the form

$$Fu = \frac{G'}{G} = \sum_\nu \frac{1}{u-a_\nu},$$

or more generally

$$\sum \frac{g_\nu u}{(u-a_\nu)\, g_\nu(a_\nu)},$$

which I shall specialize somewhat and write

$$Fu = \frac{G'}{G} = \sum \frac{u^{m_\nu}}{(u-a_\nu)\, a_\nu^{m_\nu}},$$

where to the series $a_1, a_2 \ldots a_n$ can be coordinated a second series $m_1, m_2 \ldots m_n$ such that Fu will converge for all values of the argument. I shall now form a transcendental function with the same infinite points as Fu. Taking first a single term of Fu

$$\frac{u^m}{(u-a)\,a^m}$$

this can be written in the form

$$\frac{1}{u-a}+\frac{u^m-a^m}{(u-a)\,a^m}=\frac{1}{u-a}+\frac{u^{m-1}}{a^m}+\frac{u^{m-2}}{a^{m-1}}+\ldots\frac{u^0}{a}.$$

If now we write

$$fu=\left(1-\frac{u}{a}\right)e^{\frac{u}{a}+\frac{1}{2}\frac{u^2}{a^2}+\ldots\frac{1}{m}\frac{u^m}{a^m}},$$

we shall have

$$\frac{f'u}{fu}=\frac{1}{u-a}+\frac{1}{a}+\frac{u}{a^2}+\ldots+\frac{u^{m-1}}{a^m}=\frac{u^m}{(u-a)\,a^m},$$

and the general form of the function is discovered. In order to include all the terms we will denote by

$E(u, 0)$ the function $1-u$
$E(u, 1)$ " " $(1-u)e^u$
$E(u, 2)$ " " $(1-u)e^{u+\frac{1}{2}u^2}$

. . . .

$E(u, n)$ " " $(1-u)e^{u+\frac{1}{2}u^2+\ldots+\frac{1}{n}u^n}$

whereupon the m^{th} term of Fu is

$$\frac{u^m}{(u-a)\,a^m}=\frac{E'\left(\frac{u}{a}, m\right)}{E\left(\frac{u}{a}, m\right)}=\frac{d}{du}\log E\left(\frac{u}{a}, m\right)$$

and

$$Fu=\sum_{\nu=0}^{\infty}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}}=\frac{d}{du}\log\prod_{\nu=0}^{\infty}E\left(\frac{u}{a_\nu}, m_\nu\right)=\frac{G'}{G}.$$

If zero is to be a zero point for G of the order λ, since we dare not include this value among the a's, we must write

$$G=u^\lambda\prod_{\nu=0}^{\infty}E\left(\frac{u}{a_\nu}, m_\nu\right).$$

The conditions of convergence are best discovered by separating the sum for $F(u)$ into two parts

$$Fu=\frac{G'}{G}=\sum_{\nu=0}^{m}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}}+\sum_{m+1}^{+\infty}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}},$$

and throwing the second sum into the form

$$-\sum_{\nu=m+1}^{\infty}\frac{1}{1-\frac{u}{a_\nu}}\cdot\frac{u^{m_\nu}}{a_\nu^{m_\nu+1}}=-\sum_{\nu=m+1}^{\infty}\sum_{r=0}^{\infty}\frac{u^{m_\nu+r}}{a_\nu^{m_\nu+r+1}};$$

whereupon $$G = u^{\lambda} \prod_{\nu=0}^{m} E\left(\frac{u}{a_\nu}, m_\nu\right) - \sum_{\nu=m+1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{m_\nu + r} \frac{u^{m_\nu + r}}{a_\nu^{m_\nu + r}}.$$

If now the sequence of zero-points a_ν forms only a single arithmetic series, then the uniform convergence will be assured if we assume $m_1 = m_2 = \ldots = m_n = 1$, when $$G = u^{\lambda} \prod_{\nu=0}^{\infty} \left(1 - \frac{u}{a_\nu}\right) e^{\frac{u}{a_\nu}}.$$

If the sequence of zero values extends from $-\infty$ to $+\infty$, then by multiplying together corresponding positive and negative pairs, and when $\lambda = 1$, $a_\nu = n\pi$,

$$G = u \prod_{n=-\infty}^{+\infty}{}' \left(1 - \frac{u}{n\pi}\right) e^{\frac{u}{n\pi}} = u \prod_{n=+1}^{+\infty} \left(1 - \frac{u^2}{(n\pi)^2}\right).$$

This last formula has indeed long been known, but the first form where the sine is represented as the product of "prime-functions" is the discovery of Weierstrass. In case the a_ν's form two additively independent series of zero points so that for example

$$a_\nu = w = m 2\omega + m' 2\omega'$$
$$m, m' = 0 \pm 1 \pm 2 \ldots \pm \infty,$$

the formula becomes $$G = u \prod_{\nu=-\infty}^{+\infty} E\left(\frac{u}{w}, m_\nu\right),$$

where m_ν is so to be determined that the function shall converge uniformly for all finite argument values. The convergence depends on the convergence of

$$\sum_{\nu=m+1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{m_\nu + r} \frac{u^{m_\nu} + r}{a_\nu^{m_\nu} + r}.$$

To this end it is sufficient if the convergence of

$$\sum_{\nu=m+1}^{\infty} \frac{1}{m_\nu} \cdot \text{mod.} \frac{1}{a_\nu} \cdot \text{mod.} \left(\frac{u}{a_\nu}\right)^{m_\nu}$$

is assured. Since now a_ν is a complex number it is no longer the case that $\sum_\nu \frac{1}{a_\nu^2}$ converges. It can be shown however that $\sum \frac{1}{a_\nu^{m_\nu + 1}}$ where a_ν is complex does converge for $m_\nu = 2$ or more. It is sufficient then to take $m_\nu = 2$ and the function is

$$G = u \prod_{-\infty}^{+\infty}{}' \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}} = \sigma u$$

$$w = m 2\omega + m' 2\omega'$$
$$m_1 m_1 = 0 \pm 1 \pm 2 \ldots \pm \infty$$
$$\text{except } m = m' = 0,$$

and the required function is displayed as the continued product of an infinite number of prime functions, each of which has its own polar singularity and each of which has one essential singularity at infinity.

The function σu is the simplest possible function that satisfies the condition named. Being produced in the same manner as the sine, the sigma function degrades into the sine easily at the limit,

$$\lim.\ R\left(\frac{\omega'}{\omega i}\right) = \infty \text{ (R signifies real component) when}$$

$$\lim.\ \sigma u = u\Pi'\left(1 - \frac{u}{m2\omega}\right) e^{\frac{u}{m2\omega} + \frac{1}{2}\left(\frac{u}{m2\omega}\right)^2}$$

or taking the second form (see p. 178, vol. VI),

$$\lim.\ \sigma u = u\Pi'\left(1 - \left(\frac{u}{m2\omega}\right)^2\right) e^{\left(\frac{u}{m2\omega}\right)^2}$$

$$= \frac{2\omega}{\pi} e^{\frac{1}{6}\left(\frac{u\pi}{2\omega}\right)^2} . \sin \frac{u\pi}{2\omega},$$

where the zero-points of lim. σu all lie on the real axis, and are contained in the formula $n.2\omega,\ n = 0 \pm 1,\ \pm 2,\ \pm \infty.$

A similar formula applies to a function with the zero-points represented by each of the formulæ $n.2\omega \pm 2\omega',\ n2\omega \pm 2.2\omega' \ldots$

or generally $n2\omega + n'2\omega'.$

Mnltiplying together at first those pairs which are symmetrically situated with respect to the origin, we arrive at an expression of the sigma-function as a simply infinite product

$$\sigma u = \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega} e^{\frac{1}{6}\left(\frac{u\pi}{2\omega}\right)^2} \Pi_n \left\{ \frac{\sin \frac{\pi}{2\omega}(2n\omega' - u) \sin \frac{\pi}{2\omega}(2n\omega' + u)}{\sin^2 \frac{n\omega'\pi}{\omega}} . e^{\frac{1}{\sin^2 \frac{n\omega'\pi}{\omega}} \cdot \left(\frac{u\pi}{2\omega}\right)^2} \right\}.$$

By writing

$$\eta = \frac{\pi^2}{2\omega}\left(\frac{1}{6} + \sum_n \frac{1}{\sin^2 \frac{n\omega'\pi}{\omega}}\right),$$

$$\sigma u = e^{\frac{\eta u^2}{2\omega}} \cdot \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega} \cdot \Pi_n \left(1 - \frac{\sin^2 \frac{u\pi}{2\omega}}{\sin^2 \frac{n\omega'\pi}{\omega}}\right).$$

The first, I believe, to discover that the elliptic functions could be got at by starting from the infinite product

$$\Pi\left(1 - \frac{x}{n}\right)$$

was Eisenstein in *Crelle's Journal*, Bd. XXVII, where he has two articles on this subject. He remarks that the quotient of two infinite products of the form

$$\Pi\left(1-\frac{z}{\lambda A+\lambda' A'}\right),$$

where $\lambda = 2,\ 4,\ 6\ldots,\ \lambda' = 1,\ 3,\ 5\ldots,$ gives rise to the elliptic functions. In *Crelle*, Bd. XXXV, he gives a long discussion also on the convergence of these last products. The necessity of a factor of convergence seems not to have occurred to him. He remarks that the easiest approach to the study of elliptic functions is through infinite products, they showing most plainly the periods and *also the zero and infinite points* of the function, and yet the notion of prime factors escaped him, and was first brought out by Weierstrass thirty years later. With the aid of the expression for the sine

$$\frac{\sin \pi x}{\pi} = x\Pi\left(1-\frac{x}{n}\right)e^{\frac{x}{n}},\ n = \pm 1,\ \pm 2\ldots \pm \infty$$

$$= x\Pi\left(1-\frac{x^2}{n^2}\right) \qquad n = 1,\ 2\ldots + \infty$$

and the identity $$1-\frac{x-a}{b} = \left(1+\frac{a}{b}\right)\left(1-\frac{x}{a+b}\right),$$

we arrive at the above expressions of σu as a singly infinite product of sines.

The function σu has therefore been determined by means of its zero-points and the condition that its only essential singularity is at infinity. Its zero-points are contained in the formula

$$w = m2\omega = m'\,2\omega',$$
$$m,\ m' = 0 \pm 1 \pm 2\ldots \pm \infty.$$

If then a be a non-singular argument value for the function σu, the latter is developable in the vicinage of a and of all values congruent to a in the form

$$\sigma u = (u-a)\{A_0 + A_1(u-a) + \ldots\}.$$

If, farther, ϕu be a doubly periodic function with the zero-points

$$a_1,\ a_2,\ a_3,\ \ldots a_r$$

and the (polar) infinite points

$$b_1,\ b_2,\ b_3,\ \ldots b_s$$

within each of its period parallelograms, we can then form the quotient

$$P = \frac{\sigma(u-a_1)\sigma(u-a_2)\ldots\sigma(u-a_r)}{\sigma(u-b_1)\sigma(u-b_2)\ldots\sigma(u-b_s)},$$

which has the same zero and infinite points as ϕu. If we divide ϕu by P, we shall have a function which has no zero nor infinite points except at infinity, and is therefore of the form $e^{G(x)}$ where $G(x)$ is an entire function. The function ϕu must therefore have the form

$$\phi u = \frac{\sigma(u-a_1)\,\sigma(u-a_2)\ldots\sigma(u-a_r)}{\sigma(u-b_1)\,\sigma(u-b_2)\ldots\sigma(u-b_s)}\cdot e^{G(u)}.$$

It remains to be shown, however, under what conditions such a sigma-quotient represents a doubly periodic function; in particular, to determine the form of the function Gu, and the relations connecting the a's and b's. We have

$$\sigma(u-a_\mu+2\omega) = -\,\sigma(u-a_\mu)\,e^{2\eta(u-a_\mu+\omega)},$$

the proof of which I beg to omit at present, as it would extend beyond the intended limits of this article. Therefore

$$\phi(u+2\omega) = \phi u\,.\,(-1)^{r-s}.\,e^{2(r-s)\eta(u+\omega)-2\eta\{\Sigma a_\nu-\Sigma b_\nu\}+G(u+2\omega)-G(u)},$$

and, in order that 2ω may be a period of ϕu, the exponent of e must be a multiple of πi. We may write therefore

$$G(u+2\omega) - Gu = 2\eta\,\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega)\} + k\pi i,$$

where k being an integer must be constant in order that G may be a continuous function. In like manner

$$G(u+2\omega') - Gu = 2\eta'\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega')\} + k'\pi i\,.$$

Differentiating twice,

$$G''(u+2\omega) - G''u = 0$$
$$G''(u+2\omega') - G''u = 0.$$

The function G'' must therefore be a constant, or else an exponential series, but this last it cannot be and be doubly periodic as the equations show, G is then an integral function of a degree not higher than the second; or

$$Gu = \alpha u^2 + \beta u + \gamma,$$
$$G(u+2\omega) = Gu + 4\alpha\omega\,(u+\omega) + 2\beta\omega$$
$$G(u+2\omega') = Gu + 4\alpha\omega'(u+\omega') + 2\beta\omega',$$
$$2\eta\{\Sigma a_\nu - \Sigma c_\nu - (r-s)(u+\omega)\} + k\pi i = 4\alpha\omega\,(u+\omega) + 2\beta\omega$$
$$2\eta'\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega')\} + k'\pi i = 4\alpha\omega'(u+\omega') + \alpha\beta\omega',$$

whence

$$4\alpha\omega + 2\eta\,(r-s) = 0,$$
$$4\alpha\omega' + 2\eta'(r-s) = 0.$$

Since however

$$\omega\eta' - \omega'\eta = \pm\,\frac{\pi i}{2} \gtrless 0$$

which is shown from

$$\sigma(u+2\omega+2\omega') = -\,e^{2\eta(u+\omega+2\omega')}.\,\sigma(u+2\omega') = \sigma u\,.\,e^{2\eta(u+\omega+2\omega')+2\eta'(u+\omega')}$$

and

$$\sigma(u+2\omega+2\omega') = -\,e^{2\eta'(u+\omega'+2\omega)}.\,\sigma(u+2\omega) = \sigma u\,.\,e^{2\eta(u+\omega'+2\omega)+2\eta(u+\omega)},$$

we must have

$$r-s=0,$$

whence

$$\alpha = 0,\quad r=s,\quad Gu = \beta u + \gamma,$$

and the exponent becomes

$$2\eta\{\Sigma a_\nu - \Sigma b_\nu\} - 2\beta\omega = 2k\pi i,$$

where $2k$ must be written instead of k, since now $r = s$, and therefore from

$$\phi(u + 2\omega) = \phi u . e^{2\eta\{\Sigma a_\nu - \Sigma b_\nu\} - 2\beta\omega},$$

in order that 2ω may be a period of ϕu, the exponent of e must be a multiple of $2\pi i$ instead of πi. Whence, writing

$$\Sigma a_\nu - \Sigma b_\nu = D,$$

we have

$$\eta D + k\pi i = \beta\omega,$$
$$\eta' D + k'\pi i = \beta\omega',$$
$$(\eta\omega' - \eta'\omega)\, D = (k\omega' - k'\omega)\pi i$$
$$(\eta\omega' - \eta'\omega)\, \beta \;= (k'\eta - k\eta')\pi i$$

or since

$$\eta\omega' - \eta'\omega = \pm \frac{\pi i}{2}$$
$$D = \pm\, 2\,(k\omega' - k'\omega)$$
$$\beta = \pm\, 2\,(k\eta' - k'\eta)$$

and the function ϕ becomes

$$\phi(u) = C\,\frac{\sigma(u - a_1)\,.\,.\,.\,\sigma(u - a_r)}{\sigma(u - b_1)\,.\,.\,.\,\sigma(u - a_r)}\cdot e^{(2k'\eta - 2k\eta')u},$$

where k, k' are integers, and $\eta = \frac{\sigma'\omega}{\sigma\omega}$, $\eta' = \frac{\sigma'\omega'}{\sigma\omega'}$. If now instead of any one of the values a_ν and b_ν we take one greater or less by 2ω or $2\omega'$, the exponential factor will be increased by $ce^{\pm 2\eta u}$ or $ce^{\pm 2\eta' u}$, whereby $\Sigma a_\nu - \Sigma b_\nu$ still remains a multiple of the periods and k or k' will be increased or diminished by one. We can therefore always choose such values conjugate to a_ν and b_ν that $k = k' = 0$ and $\Sigma a_\nu = \Sigma b_\nu$, and there remains as the general expression of a doubly periodic function by means of sigma quotients,

$$\phi u = C\,\frac{\sigma(u - a_1)\,.\,.\,.\,\sigma(u - a_r)}{\sigma(u - b_1)\,.\,.\,.\,\sigma(u - b_r)},$$

where

$$\Sigma a_\nu = \Sigma b_\nu .$$

As regards the degree of ϕu in $\sigma(u - a)$ it is at once evident that it must be higher than the first, for if we had

$$\phi u = C\,\frac{\sigma(u - a_1)}{\sigma(u - b_1)},$$

since $\Sigma a_1 = \Sigma b_1$, that is, $a_1 = b_1$, the expression would reduce to a constant. The simplest form possible is therefore

$$\phi u = C\,\frac{\sigma(u - a_1)\sigma(u - a_2)}{\sigma(u - b_1)\sigma(u - b_2)},$$
$$a_1 + a_2 = b_1 + b_2,$$

which form contains those functions commonly called elliptic. This form must be discussed a little more fully, for from it springs perhaps the most convenient if not the most important formula in this system. If we write $b_1 = b_2 = 0$, and $a_1 = v$, the formula becomes

$$\phi u = C \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u},$$

where v is a constant upon which C evidently depends. We are at liberty to choose C so that

$$C = \frac{1}{\sigma^2 v},$$

whereupon

$$\phi u = \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

is a doubly periodic function of either u or v, the other assuming a non-singular constant value. Now the definition of $\wp u$ in my first note was

$$\begin{aligned} \wp u &= -\frac{d^2}{du^2}\log \sigma u \\ &= \frac{1}{u^2} + \Sigma' \left\{ \left(\frac{1}{u-w}\right)^2 - \frac{1}{w^2} \right\} \end{aligned}$$

where the value $w = 0$ was omitted. Differentiating,

$$\wp' u = -\frac{2}{u^3} - 2\Sigma' \left(\frac{1}{u-w}\right)^3,$$

or including the value $w = 0$ under the Σ,

$$\wp' u = -2\Sigma \left(\frac{1}{u-w}\right)^3,$$

which is evidently periodic, and

$$\begin{aligned} \wp'(u+2\omega) &= \wp' u \\ \wp'(u+2\omega') &= \wp' u, \end{aligned}$$

and on integrating

$$\begin{aligned} \wp(u+2\omega) &= \wp u + C, \\ \wp(u+2\omega') &= \wp u + C'. \end{aligned}$$

But

$$\begin{aligned} \sigma(-u) &= -\sigma'(u), \\ \sigma'(-u) &= \sigma'(u), \end{aligned}$$

consequently

$$\frac{\sigma'(-u)}{\sigma(-u)} = -\frac{\sigma' u}{\sigma u}$$

and

$$\wp(-u) = \wp u,$$

whence

$$C = C' = 0,$$

and $\wp u$ is periodic. We have also

$$\wp u - \wp v = \Sigma \left\{ \left(\frac{1}{u-w}\right)^2 - \left(\frac{1}{v-w}\right)^2 \right\}.$$

The function $\wp u$ becomes infinite only for $u=0$ and for the congruent values. So long then as v is not given the value zero or any value congruent to zero, $\wp u - \wp v$ regarded as a function of u will have the same infinite points as $\wp u$. The zero points are $u = \pm v$, and the points congruent to these. It follows that

$$\wp u - \wp v = C \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u}$$

as a function of u alone, or

$$\wp u - \wp v = C' \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

as a function of u or v, where C' is independent of both u and v. In order to determine C', develop both sides of the equation according to powers of u and v. As given in my former note, p. 178,

$$\wp u = \frac{1}{u^2} + \ldots$$

$$\sigma u = u + \ldots,$$

so that

$$\wp u - \wp v = \frac{1}{u^2} - \frac{1}{v^2} + \ldots$$

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v} = \frac{1}{v^2} - \frac{1}{u^2} + \ldots,$$

and we must have $C' = -1$, or

$$\wp u - \wp v = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v} = -\begin{vmatrix} 1 & \wp u \\ 1 & \wp v \end{vmatrix},$$

which is the "pocket edition" of the elliptic functions mentioned.

But we can go farther and express through sigma-quotients such expressions as

$$\begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix}.$$

The infinite points (poles) of $\wp u$ and $\wp' u$ are congruent to zero, $\wp u$ becoming infinite of the second and $\wp' u$ of the third order for $u = 0 \pm w$. The determinant must therefore as a linear function of $\wp$ and $\wp'$ have also a three-fold infinite point at zero and the congruent points, therefore also three zero points to which all other vanishing points are congruent. The determinant vanishes evidently for $u=v$, $u=w$, $u=-(v+w)$, and can be expressed as follows:

$$C \frac{\sigma(u+v+w)\sigma(u-v)\sigma(v-w)}{\sigma^3 u} \times$$

$$C_1 \frac{\sigma(v+w+u)\sigma(v-w)\sigma(v-u)}{\sigma^3 v} \times$$

$$C_2 \frac{\sigma(w+u+v)\sigma(w-u)\sigma(w-v)}{\sigma^3 w},$$

where C depends on v and w, C_1 on w and u, C_2 on u and v. We may write then for the determinant

$$C'\frac{\sigma(u+v+w)\sigma(u-v)\sigma(u-w)\sigma(v-w)}{\sigma^3u.\sigma^3v.\sigma^3w}=\begin{vmatrix}1&\wp u&\wp' u\\1&\wp v&\wp' v\\1&\wp w&\wp' w\end{vmatrix}$$

where C' now depends on u, v and w.

Multiplying by u^3 and then making $u=0$, and remembering that

$$\wp' u=\frac{-2}{u^3}+\ldots,\qquad C'\frac{\sigma(v+w)\sigma(v-w)}{\sigma^2v\sigma^2w}=-2\begin{vmatrix}1&\wp v\\1&\wp w\end{vmatrix}$$

whence, comparing with the foregoing, $C'=-2$, and

$$\frac{\sigma(u+v+w)\sigma(u-v)\sigma(v-w)}{\sigma^3u\sigma^3v\sigma^3w}=-\frac{1}{2}\begin{vmatrix}1&\wp u&\wp' u\\1&\wp v&\wp' v\\1&\wp w&\wp' w\end{vmatrix},$$

and in a similar manner the determinants of higher degrees would be represented through sigma-quotients.

I will now deduce another relation which was merely dogmatically given in my first paper. Since $\wp' u$ is periodic and an odd function, we have

$$\wp'(u+\omega)=\wp'(u-\omega)=-\wp'(\omega-u),$$

and putting $u=0$ $\qquad \wp'\omega=-\wp'\omega=0,$

and likewise $\qquad \wp'\omega'=0.$

If we set now in the above determinant of the third order, $v=\omega$ and $w=\omega'$, it becomes

$$\begin{vmatrix}1&\wp u&\wp' u\\1&\wp\omega&0\\1&\wp\omega'&0\end{vmatrix}=\wp' u\begin{vmatrix}1&\wp\omega\\1&\wp\omega'\end{vmatrix}$$

and introducing the same changes in the left-hand side of the same equation we have at once

$$\wp' u=-2\frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')\sigma(\omega-\omega')}{\sigma^3u.\sigma^3\omega.\sigma^3\omega'}\cdot\frac{1}{\begin{vmatrix}1&\wp\omega\\1&\wp\omega'\end{vmatrix}}.$$

But

$$\begin{vmatrix}1&\wp\omega\\1&\wp\omega'\end{vmatrix}=\frac{\sigma(\omega+\omega')\sigma(\omega-\omega')}{\sigma^3\omega\sigma^3\omega'},$$

whence

$$\wp' u=-\wp'(-u)=-2\frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')}{\sigma^3u.\sigma\omega.\sigma\omega'.\sigma(\omega+\omega')},$$

$$(\wp' u)^2=4\frac{\sigma(u+\omega+\omega')\sigma(\omega+\omega'-u)}{\sigma^2u.\sigma^2(\omega+\omega')}\cdot\frac{\sigma(u+\omega)\sigma(u-\omega)}{\sigma^2u.\sigma^2\omega}\cdot\frac{\sigma(u+\omega')\sigma(u-\omega')}{\sigma^2u.\sigma^2\omega'},$$

that is, $\qquad =4(\wp u-\wp(\omega+\omega'))(\wp u-\wp\omega)(\wp u-\wp\omega'),$

and, writing $\wp\omega=e_1$, $\wp(\omega+\omega')=e_2$, $\wp\omega'=e_3$, this becomes

$$(\wp' u)^2=4(\wp u-e_1)(\wp u-e_2)(\wp u-e_3).$$

On writing now s for $\wp u$, this gives us of course the well-known Weierstrassian form of the elliptic integral,

$$u = \int \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}} \cdot$$

The function $(\wp' u)^2$ is thus expressed as a rational function of $\wp u$. From the definition

$$\wp u = -\frac{d^2}{du} \log \sigma u = \frac{1}{u^2} + \Sigma' \left(\frac{1}{(u-w)^2} - \frac{1}{w^2} \right) \cdot$$

But developing according to powers of u,

$$\frac{1}{u-w} = \frac{1}{w}\left(1 + \frac{u}{w} + \frac{u^2}{w^2} + \dots\right)$$

$$\frac{1}{(u-w)^2} = \frac{1}{w^2} + 2\frac{u}{w^3} + 3\frac{u^2}{w^4} + 4\frac{u^3}{w^5} + \dots$$

$\wp u = \frac{1}{u^2} + 2u\Sigma\frac{1}{w^3} + 3u^2\Sigma\frac{1}{w^4} + \dots$, and writing $\Sigma\frac{1}{w^n} = c_n$, remarking that for n odd the sum vanishes, we have

$$\wp u = \frac{1}{u^2} + 3c_4u^2 + 5c_6u^4 + \dots$$

$$(\wp u)^2 = \frac{1}{u^4}(1 + 2.3.c_4u^4 + 10c_6u^6 + \dots)$$

$$(\wp u)^3 = \frac{1}{u^6}(1 + 9c_4u^4 + 15c_6u^6 + \dots)$$

$$\wp' u = -\frac{2}{u_3} + 2.3.c_4u + 4.5c_6u^3 + 6.7.c_8u^5 + \dots$$

$$(\wp' u)^2 = \frac{1}{u_6}(4 - 24c_4u^4 - 80c_6u^6 + \dots$$

whence the expression for $(\wp' u)^2$ takes the form

$$(\wp' u)^2 = 4\wp^3 u - 60c_4\wp u - 140c_6$$

$$= 4\wp^3 u - g_1\wp^2 u - g_2\wp u - g_3,$$

where

$$g_1 = e_1 + e_2 + e_3 = 0$$

$$g_2 = -4.(e_2e_3 + e_3e_1 + e_1e_2) = 2^2.3.5\Sigma\frac{1}{w^4}$$

$$g_3 = 4.e_1.e_2.e_3 = 2^2.5.7.\Sigma\frac{1}{w^6} \cdot$$

The addition theorem for $\wp u$ may be derived from the pocket edition

$$\wp u - \wp v = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

by logarithmic differentiation, which gives

$$\frac{\sigma'(u+v)}{\sigma(u+v)} + \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' u}{\sigma u} = \frac{\wp' u}{\wp u - \wp v},$$

$$\frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' v}{\sigma v} = \frac{-\wp' v}{\wp u - \wp v},$$

from which, as the addition theorem for the function $\frac{\sigma' u}{\sigma u}$,

$$\frac{\sigma'(u \pm v)}{\sigma(u \pm v)} = \frac{\sigma' u}{\sigma u} \pm \frac{\sigma' v}{\sigma v} + \frac{1}{2}\,\frac{\wp' u \mp \wp' v}{\wp u - \wp v}.$$

It is apparent that one more differentiation gives the addition theorem for $\wp u$, which can then easily be brought into the form given in the first paper,

$$\wp(u \pm v) = \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v.$$

Weierstrass deduces the same theorem in another way which, as being characteristic, I reproduce. Equating to zero the above determinant

$$\begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix} = \wp' w(\wp v - \wp u) + \wp' v(\wp u - \wp w) + \wp' u(\wp w - \wp v) = 0,$$

we have on squaring

$$(\wp' w)^2(\wp u - \wp v)^2 - \{\wp' v(\wp u - \wp w) - \wp' u(\wp v - \wp w)\}^2 = 0,$$

or substituting the value of $(\wp' w)^2$ obtained above

$$(\wp u - \wp v)^2(4\wp^3 w - g_2 \wp w - g_3) - \{\wp' v\, \wp u - \wp' u\, \wp v - \wp w(\wp' v - \wp' u)\}^2 = 0$$

which is satisfied for $w = u,\ v,\ u + v$, that is for $\wp w = \wp u,\ \wp v,\ \wp(u + v)$. Forming now the equation

$$4(\wp u - \wp v)^2\{(s - \wp u)(s - \wp v)(s - \wp(u + v))\} = 0,$$

this is likewise satisfied for $s = \wp u,\ \wp v,\ \wp(u + v)$,
and, since the coefficients of the highest powers, $(\wp w)^3$ and s^3, are the same in each, therefore all the coefficients of the same powers of $\wp w$ and s are equal. Equating those of $(\wp w)^2$ and s^2, we have without any reduction

$$-(\wp' u - \wp' v)^2 = 4(\wp u - \wp v)^2\{-\wp u - \wp v - \wp(u - v)\},$$

whence immediately

$$\wp(u + v) = \frac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2 - \wp u - \wp v,$$

as the addition theorem for the function $\wp u$.